Applications of Derivatives

Assertion & Reason Type Questions

Directions: In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)

b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)

c. Assertion (A) is true but Reason (R) is false

d. Assertion (A) is false but Reason (R) is true

Q1.

Assertion (A): Both sin x and cos x are decreasing functions in $\left(\frac{\pi}{2}, \pi\right)$.

Reason (R): If a differentiable function decreases in (a, b), then its derivative also decreases in (a, b).

Answer : (c) Assertion (A) is true but Reason (R) is false

Q2. Assertion (A): The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$.

Reason (R): The function x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b).

Answer : (c) Assertion (A) is true but Reason (R) is false

Q3.

Assertion (A): If the function $f(x) = \dfrac{ae^x + be^{-x}}{ce^x + de^{-x}}$ is increasing function of x, then $bc > ad$.

Reason (R): A function $f(x)$ is increasing if $f'(x) > 0$ for all x.

Answer : (d) Assertion (A) is false but Reason (R) is true

Q4. Let ƒ: R → R be differentiable and strictly increasing function throughout its domain.

Assertion (A): If |ƒ(x)| is also strictly increasing function, then ƒ(x) = 0 has no real roots.

Reason (R): At ∞ or -∞, ƒ(x) may approach to 0, but cannot be equal to zero.

Answer : (a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)

Q5. Assertion (A): Let ƒ : R → R be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then, ƒ is an increasing function.

Reason (R): If ƒ' (x) < 0, then ƒ(x) is a decreasing function.

Answer : (b) Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)

Q6. Assertion (A): $f(x) = xe^{-x}$ has maximum at x = 1.

Reason (R): ƒ' (1) = 0 and ƒ'' (1) < 0.

Answer : (a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)

Q7.

Assertion (A): The graph $y = x^3 + ax^2 + bx + c$ has extremum, if $a^2 < 3b$.

Reason (R): A function, $y = f(x)$ has an extremum, if $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ for all $x \in R$.

Answer : (a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)

Q8. Let ƒ(x) be a polynomial function of degree 6 such

that $\frac{d}{dx}(f(x)) = (x-1)^3 (x-3)^2$, then

Assertion (A): $f(x)$ has a minimum at $x = 1$.

Reason (R): When $\frac{d}{dx}(f(x)) < 0, \ \forall x \in (a-h, a)$ and $\frac{d}{dx}(f(x)) > 0, \forall x \in (a, a+h)$; where '$h$' is an

infinitesimally small positive quantity, then ƒ(x) has a minimum at x = a, provided ƒ(x) is continuous at x = a. (CBSE SQP 2023-24)

Answer : (a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)

Q9. Assertion (A): Let $f(x) = 5 - 4(x-2)^{2/3}$, then at x = 2, the function f(x) attains neither least value nor greatest value.

Reason (R): x = 2 is the only critical point of ƒ(x).

Answer : (c) Assertion (A) is true but Reason (R) is false

Q10.

Assertion (A): The absolute maximum and minimum values of $f(x) = x^2\sqrt{1+x}$ in $\left[-1, \frac{1}{2}\right]$ are $\frac{\sqrt{6}}{8}$ and 0 respectively.

Reason (R): Let f be a differentiable function on I and x_0 be any interior point of I. If f attains its absolute maximum or minimum value at x_0, then $f'(x_0) = 0$.

Answer : (a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)

Q11. Assertion (A): The absolute maximum value of the function $2x^3 - 24x$ in the interval [1, 3] is 89.

Reason (R): The absolute maximum value of the function can be obtained from the value of the function at critical points and at boundary points.

Answer : (d) Assertion (A) is false but Reason (R) is true

Assertion (A) The function $f(x) = x^2 - 4x + 6$ is strictly increasing in the interval $(2, \infty)$.

Reason (R) The function $f(x) = x^2 - 4x + 6$ is strictly decreasing in the interval $(-\infty, 2)$.

Assertion (A) $y = \frac{e^x + e^{-x}}{2}$ is an increasing function on $[0, \infty)$.

Reason (R) $y = \frac{e^x - e^{-x}}{2}$ is an increasing function on $(-\infty, \infty)$.

Assertion (A) The function $f(x) = \sin x$ decreases on the interval $(0, \pi / 2)$.

Reason (R) The function $f(x) = \cos x$ decreases on the interval $(0, \pi / 2)$.

Assertion (A) The tangents to curve $y = 7x^3 + 11$ at the points, where $x = 2$ and $x = -2$ are parallel.

Reason (R) The slope of the tangents at the points, where $x = 2$ and $x = -2$, are equal.

Assertion (A) The tangent at $x = 1$ to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = -2$.

Reason (R) When a equation of a tangent solved with the curve, repeated roots are obtained at point of tangency.

Assertion (A) The equation of tangent to the curve $y^2 = 9x$ at the point $(1, 1)$ is $9x - 2y = 7$.

Reason (R) Equation of tangent is $y - y_1 = m(x - x_1)$, where m is the slope at (x_1, y_1).

Assertion (A) The equation of the normal to the curve $y^2 = 4x$ at the point $(1, 2)$ is $x + y - 3 = 0$.

Reason (R) Equation of normal is $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$.

Assertion (A) The equation of the tangent to the curve $x^{2/3} + y^{2/3} = 2$ at $(1, 1)$ is $y + x - 2 = 0$.

Reason (R) The equation of the normal to the curve $x^{2/3} + y^{2/3} = 2$ at $(1, 1)$ is $y + x = 0$.

Assertion (A) If two positive numbers are such that sum is 16 and sum of their cubes is minimum, then numbers are 8, 8.

Reason (R) If f be a function defined on an interval I and $c \in I$ and let f be twice differentiable at c, then $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$ and $f(c)$ is local minimum value of f.

The function f be given by $f(x) = 2x^3 - 6x^2 + 6x + 5$.

Assertion (A) $x = 1$ is not a point of local maxima.

Reason (R) $x = 1$ is not a point of local minima.

Assertion (A) If manufacturer can sell x items at a price of ₹ $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is ₹ $\left(\frac{x}{5} + 500\right)$. Then, the number of items he should sell to earn maximum profit is 240 items.

Reason (R) The profit for selling x items is given by $\frac{24}{5}x - \frac{x^2}{100} - 300$.

Assertion (A) $f(x) = 2x^3 - 9x^2 + 12x - 3$ is increasing outside the interval $(1, 2)$.

Reason (R) $f'(x) < 0$ for $x \in (1, 2)$.

Assertion (A) The equation of all lines having slope 0 which are tangents to the curve $y = \dfrac{1}{x^2 - 2x + 3}$, is $y = \dfrac{1}{2}$.

Reason (R) The point at which tangent to the given curve having slope 0, is $\left(1, \dfrac{1}{2}\right)$.

Assertion (A) The absolute maximum value of the function $2x^3 - 24x$ in the interval $[1, 3]$ is 89.
Reason (R) The absolute maximum value of the function can be obtained from the value of the function at critical points and at boundary points.

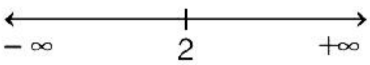
Assertion (A) If x is real, then the minimum value of $x^2 - 8x + 17$ is 1.

Reason (R) If $f''(x) > 0$ at critical point, then the value of the function at critical point will be the minimum value of the function.

12. (b) 13. (b) 14. (d)

We have, $f(x) = x^2 - 4x + 6$

or $f'(x) = 2x - 4 = 2(x - 2)$



Therefore, $f'(x) = 0$ gives $x = 2$.

Now, the point $x = 2$ divides the real line into two disjoint intervals namely, $(-\infty, 2)$ and $(2, \infty)$.

In the interval $(-\infty, 2)$, $f'(x) = 2x - 4 < 0$. Therefore, f is strictly decreasing in this interval.

Also, in the interval $(2, \infty)$, $f'(x) > 0$ and so the function f is strictly increasing in this interval.

Hence, both the statements are true but Reason is not the correct explanation of Assertion.

Assertion Let $f(x) = \dfrac{e^x + e^{-x}}{2}$

$$\Rightarrow \quad f'(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(e^x - \frac{1}{e^x}\right)$$

$$= \frac{1}{2}\left(\frac{e^{2x} - 1}{e^x}\right) \quad \ldots \text{(i)}$$

Now, for $x \geq 0$, we have

$$2x \geq 0 \Rightarrow e^{2x} \geq e^0$$

[$\because e^x$ is an increasing function]

$$\Rightarrow \quad e^{2x} \geq 1$$

Also, for $x \geq 0$

$$\Rightarrow \quad e^x \geq 1$$

$\therefore$ From Eq. (i), we have

$$f'(x) = \frac{1}{2}\left(\frac{e^{2x} - 1}{e^x}\right) \geq 0$$

So, $f(x)$ is an increasing function on $[0, \infty)$.

Reason Let $g(x) = \dfrac{e^x - e^{-x}}{2}$

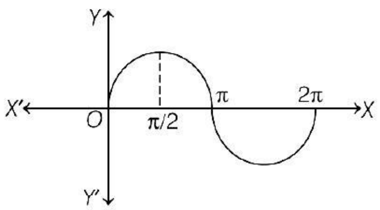
$$\Rightarrow \quad g'(x) = \frac{e^x + e^{-x}}{2} > 0$$

[$\therefore e^x$ and e^{-x} both are greater than zero in $(-\infty, \infty)$]

So, $g(x)$ is an increasing function on $(-\infty, \infty)$.

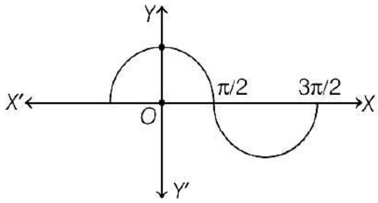
Hence, both Assertion and Reason are true.

Assertion Given, function $f(x) = \sin x$



From the graph of $\sin x$, we observe that $f(x)$ increases on the interval $(0, \pi/2)$.

Reason Given function is $f(x) = \cos x$.



From the graph of $\cos x$, we observe that, $f(x)$ decreases on the interval $(0, \pi/2)$.

Hence, Assertion is false and Reason is true.

The equation of the given curve is

$$y = 7x^3 + 11 \quad \ldots \text{(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 7 \times 3x^2 = 21x^2$$

[differentiating w.r.t. x]

$\because$ The slope of the tangent to the curve at (x_0, y_0) is $\left(\dfrac{dy}{dx}\right)_{(x_0, y_0)}$

$\therefore$ Slope of tangent at $x = 2$ is

$$\left(\frac{dy}{dx}\right)_{x=2} = 21(2)^2 = 84$$

Slope of tangent at $x = -2$ is

$$\left(\frac{dy}{dx}\right)_{x=-2} = 21(-2)^2 = 84$$

It is observed that the slopes of the tangents at the points where, $x = 2$ and $x = -2$ are equal. Hence, the two tangents are parallel.

Hence, both Assertion and Reason are true and Reason is the correct explanation of Assertion.

When $x = 1$, then $y = (1)^3 - (1)^2 - 1 + 2 = 1$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 2x - 1 \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 0$$

$\therefore$ Equation of tangent at point $(1, 1)$ is

$$y - 1 = 0(x - 1) \Rightarrow y = 1$$

Solving with the curve,

$$x^3 - x^2 - x + 2 = 1$$

$$\Rightarrow \quad x^3 - x^2 - x + 1 = 0$$

$$\Rightarrow \quad (x-1)(x^2 - 1) = 0 \Rightarrow = 1, 1, -1$$

[here, 1 is repeated root]

$\therefore$ Tangent meets the curve again at $x = -1$

$\therefore$ Assertion is false, Reason is true.

The equation of the given curve is $y^2 = 9x$

$\Rightarrow \quad y^2 = 9x$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 9 \Rightarrow \frac{dy}{dx} = \frac{9}{2y}$$

$\therefore$ Slope of tangent at (1, 1) is

$$\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{9}{2\times 1} = \frac{9}{2}$$

$$\Rightarrow \quad m = \frac{9}{2}$$

$\therefore$ Equation of tangent at (1, 1) is

$$y - 1 = \frac{9}{2}(x-1)$$

$$\Rightarrow \quad 2(y-1) = 9(x-1)$$

$$\Rightarrow \quad 2y - 2 = 9x - 9$$

$$\Rightarrow \quad 0 = 9x - 9 - 2y + 2$$

$$\Rightarrow \quad 9x - 2y - 7 = 0$$

$$\Rightarrow \quad 9x - 2y = 7$$

Hence, both Assertion and Reason are true and Reason is the correct explanation of Assertion.

The equation of the given curve is $y^2 = 4x$

On differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 4$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$\therefore$ Slope of tangent at (1, 2), is $\left(\frac{dy}{dx}\right)_{(1,\,2)} = \frac{2}{2} = 1$

Slope of normal at the point (1, 2) $= -\frac{1}{1} = -1$

$\therefore$ Equation of the normal at (1, 2) is

$$y - 2 = -1(x-1)$$

$$\Rightarrow \quad y - 2 = -x + 1$$

$$\Rightarrow \quad x + y - 3 = 0$$

So, both Assertion and Reason are true and Reason is the correct explanation of Assertion.

Assertion Differentiating $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ with respect to x, we get

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Therefore, the slope of the tangent at (1, 1) is $\left.\frac{dy}{dx}\right|_{(1,\,1)} = -1.$

So, the equation of the tangent at (1, 1) is

$$y - 1 = -1(x-1) \Rightarrow y + x - 2 = 0$$

Reason Also, the slope of the normal at (1, 1) is given by

$$\frac{-1}{\text{Slope of the tangent at }(1,\ 1)} = 1$$

Therefore, the equation of the normal at (1, 1) is

$$y - 1 = (x-1) \Rightarrow y - x = 0$$

Hence, Assertion is true and Reason is false.

Let one number be x, then the other number will be $(16 - x)$.

Let the sum of the cubes of these numbers be denoted by S.

Then, $S = x^3 + (16-x)^3$

On differentiating w.r.t. x, we get

$$\frac{dS}{dx} = 3x^2 + 3(16-x)^2(-1)$$

$$= 3x^2 - 3(16-x)^2$$

$$\Rightarrow \quad \frac{d^2S}{dx^2} = 6x + 6(16-x) = 96$$

For minima put $\frac{dS}{dx} = 0$.

$$\therefore \quad 3x^2 - 3(16-x)^2 = 0$$

$$\Rightarrow \quad x^2 - (256 + x^2 - 32x) = 0$$

$$\Rightarrow \quad 32x = 256$$

$$\Rightarrow \quad x = 8$$

At $x = 8$, $\left(\frac{d^2S}{dx^2}\right)_{x=8} = 96 > 0$

By second derivative test, $x = 8$ is the point of local minima of S.

Thus, the sum of the cubes of the numbers is the minimum when the numbers are 8 and $16 - 8 = 8$.

Hence, the required numbers are 8 and 8.

We have,

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

$$\Rightarrow \quad f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2$$

and $f''(x) = 12(x-1)$

Now, $f'(x) = 0$ gives $x = 1$.
Also, $f''(1) = 0$.
Therefore, the second derivative test fails in this case.
So, we shall go back to the first derivative test.
Using first derivatives test, we get $x = 1$ is neither a point of local maxima nor a point of local minima and so it is a point of inflexion.

Let $S(x)$ be the selling price of x items and let $C(x)$ be the cost price of x items.
Then, we have

$$S(x) = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$$

and $\quad C(x) = \frac{x}{5} + 500$

Thus, the profit function $P(x)$ is given by

$$P(x) = S(x) - C(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

i.e. $P(x) = \frac{24}{5}x - \frac{x^2}{100} - 500$

On differentiating both sides w.r.t. x, we get

$$P'(x) = \frac{24}{5} - \frac{x}{50}$$

Now, $\quad P'(x) = 0$ gives $x = 240$.

Also, $\quad P''(x) = \frac{-1}{50}$.

So, $\quad P''(240) = \frac{-1}{50} < 0$

Thus, $x = 240$ is a point of maxima.
Hence, the manufacturer can earn maximum profit, if he sells 240 items.

Assertion We have, $f(x) = 2x^3 - 9x^2 + 12x - 3$

$\Rightarrow \quad f'(x) = 6x^2 - 18x + 12$

For increasing function, $f'(x) \geq 0$

$\therefore \quad 6(x^2 - 3x + 2) \geq 0$

$\Rightarrow \quad 6(x-2)(x-1) \geq 0$

$\Rightarrow \quad x \leq 1$ and $x \geq 2$

$\therefore f(x)$ is increasing outside the interval $(1, 2)$, therefore it is true statement.

Reason Now, $f'(x) < 0$

$\Rightarrow \quad 6(x-2)(x-1) < 0$

$\Rightarrow \quad 1 < x < 2$

$\therefore$ Assertion and Reason are both true but Reason is not the correct explanation of Assertion.

The equation of the given curve is

$$y = \frac{1}{x^2 - 2x + 3} \quad \ldots\text{(i)}$$

The slope of the tangent to the given curve at any point (x, y) is given by

$$\frac{dy}{dx} = \frac{-1}{(x^2 - 2x + 3)^2} \frac{d}{dx}(x^2 - 2x + 3)$$

$$= \frac{-(2x-2)}{(x^2 - 2x + 3)^2} = \frac{-2(x-1)}{(x^2 - 2x + 3)^2}$$

For all tangents having slope 0, we must have

$\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{-2(x-1)}{(x^2 - 2x + 3)^2} = 0$

$\Rightarrow \quad -2(x-1) = 0 \Rightarrow x = 1$

From Eq. (i), we get

$$y = \frac{1}{1^2 - 2 \times 1 + 3} = \frac{1}{2}$$

$\therefore$ The equation of tangent to the given curve at point $\left(1, \frac{1}{2}\right)$ having slope $= 0$ is

$$y - \frac{1}{2} = 0(x-1) \Rightarrow y = \frac{1}{2}$$

Hence, the equation of the required line is $y = \frac{1}{2}$.

Hence, both Assertion and Reason are true.

Let $\quad f(x) = 2x^3 - 24x$

$\Rightarrow \quad f'(x) = 6x^2 - 24 = 6(x^2 - 4)$

$\quad = 6(x+2)(x-2)$

For maxima or minima put $f'(x) = 0$.

$\Rightarrow \quad 6(x+2)(x-2) = 0$

$\Rightarrow \quad x = 2, -2$

We first consider the interval $[1, 3]$.

So, we have to evaluate the value of f at the critical point $x = 2 \in [1, 3]$ and at the end points of $[1, 3]$.

At $x = 1$, $\quad f(1) = 2 \times 1^3 - 24 \times 1 = -22$

At $x = 2$, $f(2) = 2 \times 2^3 - 24 \times 2 = -32$

At $x = 3$, $f(3) = 2 \times 3^3 - 24 \times 3 = -18$

$\therefore$ The absolute maximum value of $f(x)$ in the interval $[1, 3]$ is -18 occurring at $x = 3$.

Hence, Assertion is false and Reason is true.

Let $f(x) = x^2 - 8x + 17$

$\therefore \quad f'(x) = 2x - 8$

So, $f'(x) = 0$, gives $x = 4$

Here $x = 4$ is the critical number

Now, $f''(x) = 2 > 0, \forall x$

So, $x = 4$ is the point of local minima.

$\therefore$ Minimum value of $f(x)$ at $x = 4$,

$$f(4) = 4 \times 4 - 8 \times 4 + 17 = 1$$

Hence, we can say that both Assertion and Reason are true and Reason is the correct explanation of the Assertion.

The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$ in rupees.

Assertion (A): The marginal revenue when $x = 5$ is 66.

Reason (R): Marginal revenue is the rate of change of total revenue with respect to the number of items sold at an instance.

Ans. Option (A) is correct.

Marginal revenue is the rate of change of total revenue with respect to the number of items sold at an instance. Therefore R is true.

$$R'(x) = 6x + 36$$
$$R'(5) = 66$$

$\therefore$ A is true.

R is the correct explanation of A.

The radius r of a right circular cylinder is increasing at the rate of 5 cm/min and its height h, is decreasing at the rate of 4 cm/min.

Assertion (A): When $r = 8$ cm and $h = 6$ cm, the rate of change of volume of the cylinder is 224π cm³/min

Reason (R): The volume of a cylinder is $V = \frac{1}{3}\pi r^2 h$

Ans. Option (C) is correct.

Explanation: The volume of a cylinder is $V = \pi r^2 h$. So R is false.

$$\frac{dr}{dt} = 5 \text{ cm / min}, \frac{dh}{dt} = -4 \text{ cm / min}$$
$$V = \pi r^2 h$$
$$\frac{dV}{dt} = \pi\left(r^2 \frac{dh}{dt} + 2hr\frac{dr}{dt}\right)$$
$$\frac{dV}{dt} = \pi[64 \times (-4) + 2 \times 6 \times 8 \times 5]$$
$$\left.\frac{dV}{dt}\right)_{r=8,\, h=6} = 224\pi \text{ cm}^3 / \text{min}$$

$\therefore$ Volume is increasing at the rate of 224π cm³/min.

$\therefore$ A is true.

Assertion (A): For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then at $x = 3$ the slope of curve is decreasing at 36 units/sec.

Reason (R): The slope of the curve is $\frac{dy}{dx}$.

Ans. Option (D) is correct.

Explanation: The slope of the curve $y = f(x)$ is $\frac{dy}{dx}$. R is true.

Given curve is $y = 5x - 2x^3$

or $\frac{dy}{dx} = 5 - 6x^2$

or $m = 5 - 6x^2$ $\left[\text{slope } m = \frac{dy}{dx}\right]$

$$\frac{dm}{dt} = -12x\frac{dx}{dt} = -24x$$
$$\left[\because \frac{dx}{dt} = 2 \text{ units / sec}\right]$$
$$\left.\frac{dm}{dt}\right|_{x=3} = -72$$

Rate of Change of the slope is decreasing by 72 units/s.

A is false.

A particle moves along the curve $6y = x^3 + 2$.

Assertion (A): The curve meets the Y axis at three points.

Reason (R): At the points $\left(2, \frac{5}{3}\right)$ and $(-2, -1)$ the ordinate changes two times as fast as the abscissa.

Ans. Option (D) is correct.

Explanation:

On Y axis, $x = 0$. The curve meets the Y axis at only one point, *i.e.*, $\left(0, \frac{1}{3}\right)$.

Hence A is false.

$$6y = x^3 + 2$$

or $6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$

Given, $\frac{dy}{dt} = 2\frac{dx}{dt}$

or $12 = 3x^2$

or $x = \pm 2$

Put $x = 2$ and -2 in the given equation to get y

$\therefore$ The points are $\left(2, \frac{5}{3}\right), (-2, -1)$

R is true.

Assertion (A): At $x = \frac{\pi}{6}$, the curve $y = 2\cos^2(3x)$ has a vertical tangent.

Reason (R): The slope of tangent to the curve $y = 2\cos^2(3x)$ at $x = \frac{\pi}{6}$ is zero.

Ans. Option (D) is correct.

Explanation:

Given
$$y = 2\cos^2(3x)$$
$$\frac{dy}{dx} = 2\times2\times\cos(3x)\times(-\sin 3x)\times3$$
$$\frac{dy}{dx} = -6\sin 6x$$
$$\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{6}} = -6\sin\pi = -6\times0 = 0$$

$\therefore$ R is true.

Since the slope of tangent is zero, the tangent is parallel to the X-axis. That is the curve has a horizontal tangent at $x = \frac{\pi}{6}$. Hence A is false.

Assertion (A): The equation of tangent to the curve $y = \sin x$ at the point (0, 0) is $y = x$.

Reason (R): If $y = \sin x$, then $\frac{dy}{dx}$ at $x = 0$ is 1.

Ans. Option (A) is correct.

Explanation: Given $y = \sin x$
$$\frac{dy}{dx} = \cos x$$

Slope of tangent at (0, 0) $= \left[\frac{dy}{dx}\right]_{(0,0)}$
$$= \cos 0° = 1$$

$\therefore$ R is true.

Equation of tangent at (0, 0) is
$$y - 0 = 1(x - 0)$$
$\Rightarrow$ $$y = x.$$

Hence A is true.

R is the correct explanation of A.

Assertion (A): The slope of normal to the curve $x^2 + 2y + y^2 = 0$ at (–1, 2) is –3.

Reason (R): The slope of tangent to the curve $x^2 + 2y + y^2 = 0$ at (–1, 2) is $\frac{1}{3}$.

Ans. Option (A) is correct.

Explanation:

Given
$$x^2 + 2y + y^2 = 0$$
$$2x + 2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(2+2y) = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2(1+y)} = -\frac{x}{1+y}$$

Slope of tangent at (–1, 2)
$$\left[\frac{dy}{dx}\right]_{(-1,2)} = \frac{-(-1)}{1+2} = \frac{1}{3}$$

Hence R is true.

Slope of normal at (–1, 2)
$$= \frac{-1}{\text{Slope of tangent}} = -3.$$

Hence A is true.

R is the correct explanation for A.

The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$.

Assertion (A): The value of a is ±2

Reason (R): The value of b is ±7

Ans. Option (C) is correct.

Explanation:
$$y^2 = ax^3 + b$$

Differentiate with respect to x,
$$2y\frac{dy}{dx} = 3ax^2$$

or $$\frac{dy}{dx} = \frac{3ax^2}{2y}$$

or $$\frac{dy}{dx} = \frac{3ax^2}{\pm2\sqrt{ax^3+b}} \qquad [\because y^2 = ax^3 + b]$$

or $$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3a(2)^2}{\pm2\sqrt{a(2)^3+b}} = \frac{12a}{\pm2\sqrt{8a+b}} = \frac{6a}{\pm\sqrt{8a+b}}$$

Since (2, 3) lies on the curve
$$y^2 = ax^3 + b$$
or $$9 = 8a + b \qquad \text{...(i)}$$

Also from equation of tangent
$$y = 4x - 5$$
slope of the tangent = 4

$\therefore$ $\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{6a}{\pm\sqrt{8a+b}}$ becomes

$$4 = \frac{6a}{\pm\sqrt{9}} \qquad \{\text{from (i)}\}$$

$$\therefore \quad 4 = \frac{6a}{\pm 3}$$

$$\therefore \quad 4 = \frac{6a}{3} \text{ or } 4 = \frac{6a}{-3}$$

either, $a = 2$ or $a = -2$

For $a = 2$,

$9 = 8(2) + b$

or $b = -7$

$\therefore$ $a = 2$ and $b = -7$

and for $a = -2$,

$9 = 8(-2) + b$

or $b = 25$

or $a = -2$ and $b = 25$

Hence A is true and R is false.

Assertion (A): The function $f(x) = x^3 - 3x^2 + 6x - 100$ is strictly increasing on the set of real numbers.

Reason (R): A strictly increasing function is an injective function.

Ans. Option (B) is correct.

Explanation:

$$f(x) = x^3 - 3x^2 + 6x - 100$$
$$f'(x) = 3x^2 - 6x + 6$$
$$= 3[x^2 - 2x + 2]$$
$$= 3[(x-1)^2 + 1]$$

since $f'(x) > 0;\ x \in R$

$f(x)$ is strictly increasing on R.

Hence A is true.

For a strictly increasing function,

$x_1 > x_2$

$\Rightarrow$ $f(x_1) > f(x_2)$

i.e.; $x_1 = x_2$

$\Rightarrow$ $f(x_1) = f(x_2)$

Hence, a strictly increasing function is always an injective function.

So R is true.

But R is not the correct explanation of A.

Consider the function $f(x) = \sin^4 x + \cos^4 x$.

Assertion (A): $f(x)$ is increasing in $\left[0, \frac{\pi}{4}\right]$

Reason (R): $f(x)$ is decreasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

Ans. Option (B) is correct.

Explanation:

$f(x) = \sin^4 x + \cos^4 x$

or $f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$= -4\sin x\cos x\,[-\sin^2 x + \cos^2 x]$

$= -2\sin 2x \cos 2x$

$= -\sin 4x$

On equating,

$f'(x) = 0$

or $-\sin 4x = 0$

or $4x = 0, \pi, 2\pi, \ldots\ldots\ldots$

or $x = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

Sub-intervals are $\left[0, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

or $f'(x) < 0$ in $\left[0, \frac{\pi}{4}\right]$

or $f(x)$ is decreasing in $\left[0, \frac{\pi}{4}\right]$

and, $f'(x) > 0$ in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

$\therefore f'(x)$ is increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Both A and R are true. But R is not the correct explanation of A.

Assertion (A): The function $y = [x(x-2)]^2$ is increasing in $(0, 1) \cup (2, \infty)$

Reason (R): $\frac{dy}{dx} = 0$, when $x = 0, 1, 2$.

Ans. Option (B) is correct.

Explanation:

$$y = [x(x-2)]^2$$
$$= [x^2 - 2x]^2$$

$\therefore$ $\frac{dy}{dx} = 2(x^2 - 2x)(2x - 2)$

or $\frac{dy}{dx} = 4x(x-1)(x-2)$

On equating $\frac{dy}{dx} = 0$,

$4x(x-1)(x-2) = 0 \Rightarrow x = 0, x = 1, x = 2$

$\therefore$ Intervals are $(-\infty, 0), (0,1), (1,2), (2,\infty)$

Since, $\frac{dy}{dx} > 0$ in $(0,1)$ or $(2, \infty)$

$\therefore f(x)$ is increasing in $(0,1) \cup (2, \infty)$

Both A and R are true. But R is not the correct explanation of A.

Assertion (A): The function $y = \log(1 + x) - \frac{2x}{2+x}$ is a decreasing function of x throughout its domain.

Reason (R): The domain of the function $f(x) = \log(1 + x) - \frac{2x}{2+x}$ is $(-1, \infty)$

Ans. Option (D) is correct.

Explanation:

$\log(1 + x)$ is defined only when $x + 1 > 0$ or $x > -1$.

Hence R is true.

$$y = \log(1+x) - \frac{2x}{2+x}$$

Diff. w.r.t. 'x',

$$\frac{dy}{dx} = \frac{1}{1+x} - \frac{[(2+x)(2)-2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{[4-2x-2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2-4(1+x)}{(2+x)^2(1+x)}$$

$$= \frac{4+x^2+4x-4-4x}{(2+x)^2(1+x)}$$

$$= \frac{x^2}{(2+x)^2(1+x)}$$

For increasing function,

$$\frac{dy}{dx} \geq 0$$

or $$\frac{x^2}{(2+x)^2(x+1)} \geq 0$$

or $$\frac{(2+x)^2(x+1)x^2}{(2+x)^4(x+1)^2} \geq 0$$

or $$(2+x)^2(x+1)x^2 \geq 0$$

When $x > -1$,

$\frac{dy}{dx}$ is always greater than zero.

$\therefore$ $$y = \log(1+x) - \frac{2x}{2+x}$$

is always increasing throughout its domain.

Hence A is false.

The sum of surface areas (S) of a sphere of radius 'r' and a cuboid with sides $\frac{x}{3}$, x and $2x$ is a constant.

Assertion (A): The sum of their volumes (V) is minimum when x equals three times the radius of the sphere.

Reason (R): V is minimum when $r = \sqrt{\frac{S}{54+4\pi}}$

Ans. Option (A) is correct.

Explanation:

Given $$S = 4\pi r^2 + 2\left[\frac{x^2}{3} + 2x^2 + \frac{2x^2}{3}\right]$$

$$S = 4\pi r^2 + 6x^2$$

or $$x^2 = \frac{S-4\pi r^2}{6}$$

and $$V = \frac{4}{3}\pi r^3 + \frac{2x^3}{3}$$

$\therefore$ $$V = \frac{4}{3}\pi r^3 + \frac{2}{3}\left(\frac{S-4\pi r^2}{6}\right)^{3/2}$$

$$\frac{dV}{dr} = 4\pi r^2 + \left(\frac{S-4\pi r^2}{6}\right)^{1/2}\left(\frac{-8\pi r}{6}\right)$$

$$\frac{dV}{dr} = 0$$

or $$r = \sqrt{\frac{S}{54+4\pi}}$$

Now $$\frac{d^2V}{dr^2} = 8\pi r + \left(\frac{-8\pi}{6}\right)\left(\frac{S-4\pi r^2}{6}\right)^{1/2} + \frac{1}{2}\left(\frac{S-4\pi r^2}{6}\right)^{-1/2}\cdot\left(\frac{-8\pi r}{6}\right)$$

at $$r = \sqrt{\frac{S}{54+4\pi}};\ \frac{d^2V}{dr^2} > 0$$

$\therefore$ for $r = \sqrt{\frac{S}{54+4\pi}}$ volume is minimum

i.e., $r^2(54 + 4\pi) = S$

or $r^2(54 + 4\pi) = 4\pi r^2 + 6x^2$

or $6x^2 = 54r^2$

or $x^2 = 9r^2$

or $x = 3r$

Hence both A and R are true.

R is the correct explanation of A.

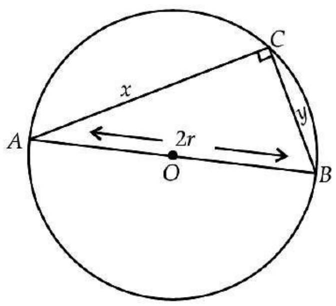
AB is the diameter of a circle and C is any point on the circle.

Assertion (A): The area of ΔABC is maximum when it is isosceles.

Reason (R): ΔABC is a right-angled triangle.

Ans. Option (A) is correct.

Explanation:



Let the sides of rt. $\Delta\, ABC$ be x and y.

$\therefore$ $$x^2 + y^2 = 4r^2$$

and $$A = \text{Area of } \Delta = \frac{1}{2}xy$$

Let, $$S = A^2$$

$$= \frac{1}{4}x^2y^2$$

$$= \frac{1}{4}x^2(4r^2 - x^2)$$

$$= \frac{1}{4}(4r^2x^2 - x^4)$$

$\therefore$ $$\frac{dS}{dx} = \frac{1}{4}[8r^2x - 4x^3]$$

or $$\frac{dS}{dx} = 0$$

or $$x^2 = 2r^2 \text{ or } x = \sqrt{2}r$$

and $$y^2 = 4r^2 - 2r^2 = 2r^2$$

or $$y = \sqrt{2}r$$

i.e., $$x = y \text{ and } \frac{d^2S}{dx^2} = (2r^2 - 3x^2)$$

$$= 2r^2 - 6r^2 < 0$$

or Area is maximum, when Δ is isosceles.

Hence A is true.

Angle in a semicircle is a right angle.

$\therefore \angle C = 90°$

$\Rightarrow \Delta ABC$ is a right-angled triangle.

$\therefore$ R is true.

R is the correct explanation of A.

A cylinder is inscribed in a sphere of radius R.

Assertion (A): Height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$ units.

Reason (R): The maximum volume of the cylinder is $\frac{4\pi R^3}{\sqrt{3}}$ cubic units.

Ans. Option (C) is correct.

Explanation: Let the radius and height of cylinder be r and h respectively

$\therefore$ $$V = \pi r^2 h \quad ...(i)$$

But $$r^2 = R^2 - \frac{h^2}{4}$$

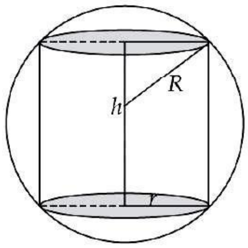
$\therefore$ $$\pi h\left(R^2 - \frac{h^2}{4}\right) = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

or $$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$$

For maximum or minimum

$\therefore$ $$\frac{dV}{dh} = 0 \text{ or } h^2 = \frac{4R^2}{3}$$

or $$h = \frac{2R}{\sqrt{3}}$$



and $$\frac{d^2V}{dh^2} = \pi\left(-\frac{6h}{4}\right) < 0$$

$$\text{Maximum volume} = \pi\cdot\left[R^2\cdot\frac{2R}{\sqrt{3}} - \frac{1}{4}\left(\frac{2R}{\sqrt{3}}\right)^3\right]$$

$$= \frac{4\pi R^3}{3\sqrt{3}} \text{ cubic units}$$

Hence A is true and R is false.

Assertion (A): The altitude of the cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Reason (R): The maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

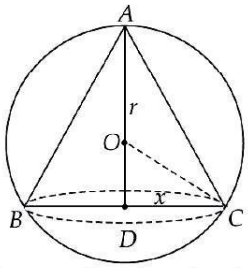
Ans. Option (B) is correct.

Explanation: Let radius of cone be x and its height be h.

$\therefore$ $$OD = (h - r)$$

Volume of cone

$$(V) = \frac{1}{3}\pi x^2 h \quad ...(i)$$



In ΔOCD, $x^2 + (h - r)^2 = r^2$ or $x^2 = r^2 - (h - r)^2$

$\therefore$ $$V = \frac{1}{3}\pi h\{r^2 - (h - r)^2\}$$

$$= \frac{1}{3}\pi(-h^3 + 2h^2 r)$$

or $$\frac{dV}{dh} = \frac{\pi}{3}(-3h^2 + 4hr)$$

$\therefore$ $$\frac{dV}{dh} = 0 \text{ or } h = \frac{4r}{3}$$

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h+4r)$$
$$= \frac{\pi}{3}\left(-6\left(\frac{4r}{3}\right)+4r\right)$$
$$= -\frac{4\pi r}{3} < 0$$

$\therefore$ at $h = \frac{4r}{3}$, Volume is maximum

Maximum volume

$$= \frac{1}{3}\pi\cdot\left\{-\left(\frac{4r}{3}\right)^3 + 2\left(\frac{4r}{3}\right)^2 r\right\}$$
$$= \frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$$
$$= \frac{8}{27}\text{ (volume of sphere)}$$

Hence both A and R are true.
R is not the correct explanation of A.

Reason (R) : Rate of change of area of a circle with respect to its radius r is $\frac{dA}{dr}$, where A is the area of the circle.

Assertion (A) : $f(x) = \tan x - x$ always increases.

Reason (R) : Any function $y = f(x)$ is increasing if $\frac{dy}{dx} > 0$.

Assertion (A) : $f(x) = x^4$ is decreasing in the interval $(0, \infty)$.

Reason (R) : Any function $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$.

Assertion (A) : The slope of the tangent to the curve $y = x^3$ where it cuts x-axis, is 0.